Equivariant homotopy commutativity for $G = C_{pqr}$

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Abstract

We investigate the combinatorial data arising from the classification of equivariant homotopy commutativity for cyclic groups of order $G = C_{p_1 \cdots p_n}$ for p_i distinct primes. In particular, we will prove a structural result which allows us to enumerate the number of N_∞-operads for C_{pqr}, verifying a computational result.

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1 Introduction

A (symmetric topological) operad is a sequence of spaces $\mathcal{O}(n)$ for $n \geq 0$, equipped with an action of the symmetric group Σ_n and compatibility conditions, where $\mathcal{O}(n)$ encodes the possibilities of n-ary operations. An $\mathcal{O}$-algebra is then a topological space X together with maps

$$\mathcal{O}(n) \times_{\Sigma_n} X^n \longrightarrow X$$

plus compatibility conditions. If $\mathcal{O}(n)$ is Σ_n-contractible for each n, we speak of an *E_∞-operad.* This type of operad governs homotopy commutativity, as the contractibility implies that all the different choices of multiplying n elements are homotopic. There are many different E_∞-operads which all have their own technical advantages, but as the homotopy theory of operads depends on the homotopy type of the underlying spaces, all E_∞-operads are equivalent in this sense, meaning that there is one notion of homotopy commutativity.

Equivariantly, this is a different story. If we move on from spaces to G-spaces for a finite group G, we do not just have Σ_n acting on X^n in the usual way but we also have to consider G permuting the factors of any product indexed over G-sets. This G-action needs to be compatible with the Σ_n-action. This then leads to the notion of *N_∞-operads.* Unlike in the nonequivariant case, not all N_∞-operads are weakly equivalent to each other. Instead, those equivalence classes are determined by so-called *transfer systems*, which are combinatorial data consisting of pairs of subgroups of G satisfying some conditions. Conversely, every such transfer system also determines an N_∞-operad. In particular, transfer systems can be depicted as graphs satisfying certain conditions, which we call an *N_∞-diagram.*

It then becomes an intriguing question to see how many different types of equivariant homotopy commutativity are possible for a finite group G, and how these are related. For a cyclic group of order p^{n-1}, an answer was given in [BBR19], namely, the number of N_∞-diagrams for $C_{p^{n-1}}$ is the n^{th} Catalan number. Moreover, the set of N_∞-diagrams for a fixed group is a lattice, which in the case of $G = C_{p^{n-1}}$ is isomorphic to the n-Tamari lattice (the vertex set of the n-associahedron).

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Therefore, types of equivariant homotopy commutativity have interesting links with well-studied combinatorial objects.

Having thus covered cyclic groups of order equal to a prime power, one might be tempted to think that there would be a similarly neat answer for any finite abelian group. However, one would soon find out that this is not the case as strange "mixed" diagrams appear whenever one considers products of groups. For example, for C_p there are two possible N_∞-diagrams, for C_{pq}, $p \neq q$, there are ten, and we will show that for C_{pqr} for distinct primes p, q and r there are 450. We will also explain why the number grows very rapidly for $C_{p_1 \cdots p_n}$ and present some structural insights into the general case.

Below we outline the main result of the paper, which gives a structural result on the collection of N_∞-operads, and suggests a more economical method to compute them.

Theorem 1.1. The set $\mathcal{N}_n$ of N_∞-diagrams for $G = C_{p_1 \cdots p_n}$ admits a decomposition into $(n+1)$ disjoint subsets

$$\mathcal{N}_n = \bigsqcup_{d=0}^{n} \mathrm{Comp}_d(G).$$

Moreover, there is an involution Φ_n on $\mathcal{N}_n$ such that

$$\Phi_n(\mathrm{Comp}_d(G)) = \mathrm{Comp}_{n-d}(G)$$

for any $0 \leq d \leq n$. In particular, we have

$$|\,\mathrm{Comp}_d(G)| = |\,\mathrm{Comp}_{n-d}(G)|.$$

2 N_∞-operads and N_∞-diagrams

Given a topological space X equipped with a multiplication $m : X \times X \longrightarrow X$, we would like to say that its multiplication is *homotopy commutative* if the diagram

$$\begin{tikzcd} X \times X \arrow[r, "m"] \arrow[d, "\tau"'] & X \\ X \times X \arrow[ur, "m"'] & \end{tikzcd}$$

commutes up to homotopy, where τ is the twist map permuting the two factors. With this in place one would now have to take care of coherence, that is, the chosen homotopy between m and $m \circ \tau$ needs to be compatible with multiplying three or more copies of X. Such coherence issues are neatly packaged in the theory of operads [May72].

Definition 2.1. A (topological) *symmetric operad* is a collection $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 0}$ of topological spaces $\mathcal{O}(n)$ equipped with a (right) Σ_n-action together with maps

$$\mathcal{O}(n) \times \mathcal{O}(i_1) \times \cdots \times \mathcal{O}(i_n) \longrightarrow \mathcal{O}(i_1 + \cdots + i_n)$$

such that the expected coherence diagrams hold with regards to associativity, unitality and the symmetric group actions.

An *algebra over an operad* $\mathcal{O}$ is a space X together with multiplication maps

$$\mathcal{O}(n) \times_{\Sigma_n} X^n \longrightarrow X$$

satisfying the expected coherence diagrams. Here, Σ_n acts on X^n by permuting factors. This information is equivalent to a morphism of symmetric operads $\mathcal{O} \longrightarrow \mathrm{End}(X)$ where $\mathrm{End}(X)$ denotes the endomorphism operad of X, i.e.,

$$\mathrm{End}(X)(n) = \mathrm{Hom}(X^n, X).$$

We can think of the space $\mathcal{O}(n)$ as the different possibilities of multiplying n elements in our space. For instance, if $\mathcal{O}(n) = *$ for all n, then there is a unique way of multiplying n elements

$$* \times_{\Sigma_n} X^n \cong X^n/\Sigma_n \longrightarrow X.$$

In particular, this means that our space X is a strictly commutative object.

If instead we suppose that $\mathcal{O}(n) \simeq *$ for all n, then there is one way of multiplying n elements "up to homotopy", which leads to the following definition.

Definition 2.2. An E_∞*-operad* is a symmetric operad $\mathcal{O}$ such that the action of Σ_n on each space is free, and every $\mathcal{O}(n)$ is Σ_n-equivariantly contractible.

There are many different E_∞-operads, each of them having their own technical advantages and disadvantages. Thankfully, all E_∞-operads are weakly equivalent, indeed, there is a Quillen model structure on the category of topological symmetric operads where the weak equivalences are those maps that are levelwise homotopy equivalences of spaces [BM03]. In particular, we can think of this as having one unique (up to homotopy) notion of homotopy commutativity.

Now that we have outlined the theory of homotopy commutativity in the non-equivariant case, we move towards to the more complex setting of G-spaces for G some finite group. We now need to consider multiplication maps of the form

$$\prod_T X \longrightarrow X$$

where T is a G-set with $n = |T|$ elements. The G-action induces a group homomorphism $G \to \Sigma_n$. This means that the $\mathcal{O}(n)$ spaces should not be thought of merely as Σ_n-spaces, but as $(G \times \Sigma_n)$-spaces. Note that simply putting a trivial G-action on the $\mathcal{O}(n)$ would not allow for multiplications of the above kind for any T with more than one element.

We shall now work towards the theory of N_∞-operads, which allows us to fix this issue.

Definition 2.3. A *graph subgroup* Γ of $G \times \Sigma_n$ is a subgroup such that $\Gamma \cap (1 \times \Sigma_n)$ is trivial. (Here 1 denotes the trivial group.)

Any graph subgroup is of the form

$$\Gamma = \{(h, \sigma(h)) \mid h \in H\},$$

with $H \leq G$ and $\sigma \colon H \longrightarrow \Sigma_n$ a group homomorphism. Moreover, given a finite H-set T with n elements we obtain a graph subgroup

$$\Gamma(T) = \{(h, \sigma(h)) \mid h \in H\},$$

where $\sigma\colon H \longrightarrow \Sigma_n$ represents the H-action on T. Conversely, we can view any graph subgroup as one of the form $\Gamma(T)$, as for

$$\Gamma = \{(h, \sigma(h)) \mid h \in H\},$$

we can set T to be a set of n elements with the H-action given by σ.

Definition 2.4. An N_∞*-operad* is a symmetric operad $\mathcal{O}$ in the category of G-spaces (that is, a collection of $G \times \Sigma_n$-spaces $\mathcal{O}(n), n \geq 0$) satisfying the following conditions.

- For all $n \geq 0$, $\mathcal{O}(n)$ is Σ_n-free.
- For every graph subgroup Γ of $G \times \Sigma_n$, the space $\mathcal{O}(n)^\Gamma$ is either empty or contractible.
- $\mathcal{O}(0)^G$ and $\mathcal{O}(2)^G$ are both nonempty.

The last condition ensures that the operad possesses an equivariant multiplication and an equivariant 'point'.

The second point together with the operad structure implies that each $\mathcal{O}(n)$ is a classifying space for a family of subgroups which satisfy some further properties forced by operad structure. This information can be distilled into the theorem below.

Theorem 2.5. Up to weak equivalence, every N_∞-operad determines and is determined by a set $X = \{N_K^H\}$, where $K < H$ are subgroups of G, satisfying the following properties and their conjugacies.

- (Transitivity) If $N_K^H \in X$ and $N_H^L \in X$, then $N_K^L \in X$.
- (Restriction) If $N_K^H \in X$ and $L \leq G$, then $N_{K\cap L}^{H \cap L} \in X$.

We will call such a set a *transfer system* and the objects N_K^H will sometimes be referred to as *norm maps.*

Blumberg and Hill showed that every operad determines an "indexing system" [BH15]. Rubin [Rub17], Gutierrez-White [GW18] and Bonventre-Pereira [BP17] independently showed that for every such indexing system one can construct a corresponding operad. Further, Barnes-Balchin-Roitzheim [BBR19] showed that indexing systems are equivalent to the transfer systems given in the above version of this theorem.

Corollary 2.6. There are as many homotopy types of N_∞-operads for a fixed finite group G as there are transfer systems for G. Q.E.D.

In particular, there can be only finitely many N_∞-operads for a finite group G, and as such, it makes sense to count them. We will denote by $N_\infty(G)$ the set of all N_∞-operads on G. For $G = C_{p^n}$, the number of N_∞-operads plus some additional structure has been determined in [BBR19].

Before continuing, let us assess the first non-trivial case. We will choose to display indexing systems as graphs whose vertices are the subgroups of G, and there is an edge $H \to K$ if $N_H^K \in X$.

Example 2.7. Let $G = C_p$ for some prime p, then there are two N_∞-operads which have the following graph representations.

$$\left(C_{p^0} \qquad\qquad C_{p^1} \right) \qquad \left(C_{p^0} \longrightarrow C_{p^1} \right)$$

The key ingredient in the result of [BBR19] is an operation

$$\odot\colon N_\infty(C_{p^i}) \times N_\infty(C_{p^j}) \to N_\infty(C_{p^{i+j+2}}).$$

In particular it is proved that every N_∞-operad for $C_{p^{i+j+2}}$ is of the form $X \odot Y$ for $X \in N_\infty(C_{p^i})$ and $Y \in N_\infty(C_{p^j})$. This then allows an inductive strategy of proof of the main result.

Theorem 2.8 ([BBR19, Theorem 1])**.** For $n \geq 1$ we have

$$|N_\infty(C_{p^n})| = \mathsf{Cat}(n+1)$$

where $\mathsf{Cat}(n)$ is the n^{th} Catalan number.

The result goes a bit further than just an enumeration result. We can put a partial order on the set of all N_∞-diagrams for a fixed group by saying that one N_∞-operad X is smaller than another N_∞-operad Y if it is a subset of Y. On the other side, there is a wealth of objects enumerated by Catalan numbers. One of them is the set of rooted binary trees. We can put a partial order on the set of rooted binary trees with n leaves by saying that one tree is larger than another if it can be obtained from the latter by rotating a branch to the right. Balchin-Barnes-Roitzheim found that, indeed, N_∞-diagrams for $G = C_{p^n}$ and rooted binary trees with $n+2$ leaves are isomorphic as posets [BBR19]. However, we do not wish to elaborate on this result here.

The goal of this paper is to study the set $N_\infty(G)$ for G a group of the form $C_{p_1 \cdots p_n}$ for p_i distinct primes where the situation is somewhat more complicated. Note that, in particular, the subgroup lattice is an n-dimensional cube.

3 Classifying N_∞-operads for $G = C_{p_1 p_2}$

In this section we explore the structure of N_∞-operads for $G = C_{p_1 p_2}$ as it will illuminate the theory that we present in the rest of the paper. Specifically, it is the only non-trivial case where one can visualise the entire situation. One can check that there are ten such N_∞-operads as follows.

$$\begin{tikzcd} 1 & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[d] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] \arrow[d] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[d] & C_{p_1} \arrow[d] \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \arrow[d] \\ C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \\ C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \arrow[d] \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] & C_{p_1} \\ C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd}$$

FIGURE 1. The ten possible N_∞-operad structures for $G = C_{p_1 p_2}$.

Note that there is an odd one out in these diagrams, namely the following diagram, which has a diagonal which is not forced by the transitivity rule. It is this type of N_∞-operad, which we call a "mixed diagram", that causes the complexity in this problem.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}$$

Figure 2. The *mixed* N_∞-operad for $G = C_{p_1p_2}$.

Now, we will explore the structure of this collection of ten operads, which we denote by $\mathcal{N}_2$. Let us consider three subsets of $\mathcal{N}_2$. First, denote by $\mathrm{Comp}_2(C_{p_1p_2})$ those N_∞-operads which contain the norm map $N_1^{C_{p_1p_2}}$, i.e. the diagonals in Fig. 1.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \arrow[d] \\ C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \\ C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \arrow[d] \\ C_{p_2} & C_{p_1p_2} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}$$

Figure 3. The collection $\mathrm{Comp}_2(C_{p_1p_2})$.

Next, we consider the collection $\mathrm{Comp}_0(C_{p_1p_2})$ of those N_∞-operads which do not contain the norm maps $N_H^{C_{p_1p_2}}$ for H any of the two proper subgroups of $C_{p_1p_2}$.

$$\begin{tikzcd} 1 & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[d] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] & C_{p_1} \\ C_{p_2} & C_{p_1p_2} \end{tikzcd}$$

Figure 4. The collection $\mathrm{Comp}_0(C_{p_1p_2})$.

We then define $\mathrm{Comp}_1(C_{p_1p_2})$ to consist of those N_∞-operads which have the norm map $N_{C_{p_1}}^{C_{p_1p_2}}$ or $N_{C_{p_2}}^{C_{p_1p_2}}$ but not $N_1^{C_{p_1p_2}}$.

$$\begin{tikzcd} 1 \arrow[d] & C_{p_1} \arrow[d] \\ C_{p_2} & C_{p_1p_2} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & C_{p_1} \\ C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd}$$

Figure 5. The collection $\mathrm{Comp}_1(C_{p_1p_2})$.

We can see, purely by inspection, that these three $(= 2 + 1)$ subsets form a partition of $\mathcal{N}_2$. This will be the first part of the general strategy for understanding $\mathcal{N}_n$. We will prove that we can partition the set $\mathcal{N}_n$ into $(n + 1)$ disjoint subsets.

However, there is still some extra structure on the collection $\{\mathrm{Comp}_i(C_{p_1p_2})\}_i$, which we will now investigate. Indeed, it is no coincidence that $|\,\mathrm{Comp}_0(C_{p_1p_2})| = |\,\mathrm{Comp}_2(C_{p_1p_2})|$. In particular, there exists an involution $\Phi_2 \colon \mathcal{N}_2 \to \mathcal{N}_2$. Instead of giving a formal definition (which will appear in the following section) we simply illustrate it by giving the correspondence and inviting the reader to understand the relationship in Figure 6. We have grouped the elements in coloured blocks to distinguish the $\mathrm{Comp}_i(C_{p_1p_2})$. Note that $\Phi_2\colon\ \mathrm{Comp}_i(C_{p_1p_2}) \to \mathrm{Comp}_{2-i}(C_{p_1p_2})$. Also observe that the long diagonal is affected only in the case where $i = 0, 2$.

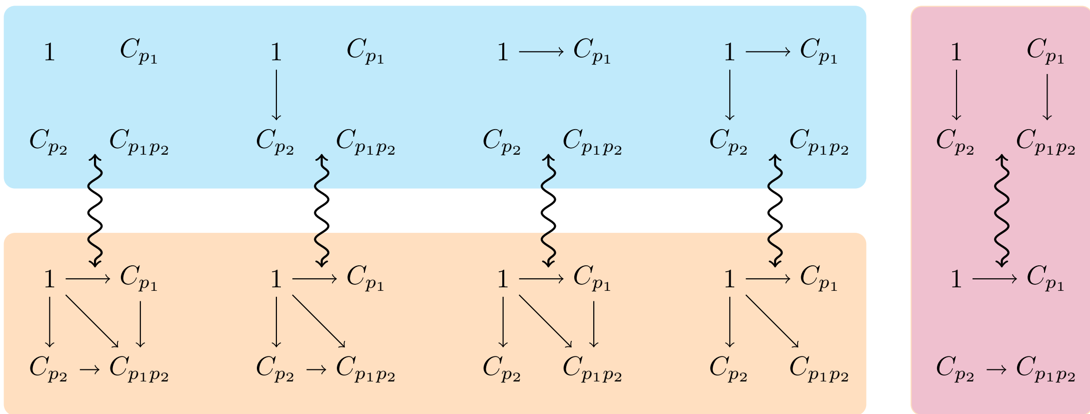

FIGURE 6. The involution $\Phi_2 \colon \mathcal{N}_2 \to \mathcal{N}_2$.

4 Higher dimensions

We now introduce the general theory of N_∞-diagrams for $G = C_{p_1\cdots p_n}$, where $p_1, \ldots, p_n$ are distinct primes. We will denote by $\mathcal{N}_n$ the set of N_∞-operads for such a G. The goal of this section is to prove that there is an intuitive decomposition of $\mathcal{N}_n$ into $(n+1)$ disjoint subsets, which, as hinted at in Section 3, admits an involution $\Phi_n \colon \mathcal{N}_n \to \mathcal{N}_n$. We then use this result in Section 5 to prove that $|\mathcal{N}_3| = 450$.

4.1 Decomposition

Definition 4.1. Let D be an N_∞-diagram for G. If $H < K$ are subgroups of G we denote by D_H^K the graph induced by D on the vertices corresponding to subgroups of K containing H.

Example 4.2. If we have $G = C_{p_1p_2}$ and

$$D = \left(\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & C_{p_1} \\ C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd} \right)$$

then $D_{C_{p_2}}^{C_{p_1}C_{p_2}} = \left(\begin{tikzcd} C_{p_2} \arrow[r] & C_{p_1p_2} \end{tikzcd} \right)$.

The set of N_∞-diagrams for $G = C_{p_1 \cdots p_n}$ admits a decomposition into $(n+1)$ disjoint subsets as follows. Let $D \in \mathcal{N}_n$, and consider the set of all arrows $(H \to G)$ contained in D. Let G^0 be the intersection of all the initial vertices of such arrows; clearly, the induced diagram $D^0 := D^G_{G^0}$ contains all the arrows with final vertex G, and it is minimal for this property. (Note that in the case of there being no arrows ending in G, we have $G^0 = G$.) Therefore, denoting by $\mathrm{Comp}_d(G)$ the set of diagrams G with $G^0 = C_{p_{i_1} p_{i_2} \cdots p_{i_{n-d}}}$ for any $0 \leq d \leq n$, we obtain a decomposition:

$$\mathcal{N}_n = \bigsqcup_{d=0}^{n} \mathrm{Comp}_d(G).$$

Example 4.3. In Figure 6, the blue (top left) box of diagrams is $\mathrm{Comp}_0(C_{p_1p_2})$, the yellow (bottom left) box is $\mathrm{Comp}_2(C_{p_1p_2})$, and the pink (rightmost) box is $\mathrm{Comp}_1(C_{p_1p_2})$.

Remark 4.4. Note that by restriction and composition, if D is in $\mathrm{Comp}_d(G)$ then D^0 is supported on (and contains the big diagonal of) a d-dimensional face, namely the arrow from $G^0 = C_{p_{i_1} p_{i_2} \cdots p_{i_{n-d}}}$ to G. In particular we see that $\mathrm{Comp}_0(n)$ consists of those N_∞-diagrams which do not contain a norm map to the group itself, and $\mathrm{Comp}_n(G)$ consists of those N_∞-diagrams which contain the long diagonal N_1^G.

To prove the next result we need to introduce some notation for the facets (codimension one faces) of the n-dimensional cube. Any facet has one of the following forms:

- *bottom facet* B_i, i.e., facet containing the 1 vertex and a vertex of the form $C_{p_1 \cdots \hat{p}_i \cdots p_n}$ ($\hat{p}_i$ means removing p_i),
- *top facet* T_i, i.e., facet containing a vertex of the form C_{p_i} and the G vertex.

Proposition 4.5. If $D \in \mathrm{Comp}_d(G)$, then there exist $n-d$ facets adjacent to 1 and not intersecting D^0 such that all arrows of D are either arrows of D^0 or contained in these $n-d$ facets.

Proof. First notice that if $D \in \mathrm{Comp}_n(G)$, then $D^0 = D$ so the result is trivially true. Now assume $d < n$ and $D \in \mathrm{Comp}_d(G)$. Clearly, as D^0 contains a diagonal of a d-dimensional face ending in G, D^0 is then contained in the intersection of $n-d$ top facets, say $T_1, T_2, \ldots, T_{n-d}$. This implies that the intersection G^0 of all initial vertices of arrows with final vertex G is given by

$$G^0 = C_{p_1 p_2 \cdots p_{n-d}}.$$

Let us show that D^0 must include an arrow $(G^0 \to G)$. To do this let us enumerate the set of arrows of D^0 with final vertex G:

$$(H_1 \to G), (H_2 \to G), \ldots, (H_k \to G).$$

Since D contains the arrows $(H_1 \to G)$ and $(H_2 \to G)$, by the restriction condition it must have an arrow $(H_1 \cap H_2 \to H_1)$. But by the transitivity condition, the arrows

$$(H_1 \cap H_2 \to H_1) \text{ and } (H_1 \to G)$$

imply the existence of an arrow $(H_1 \cap H_2 \to G)$. Repeating this argument, since $G^0 = \bigcap_{i=1}^k H_i$, we deduce that D^0 has to contain the arrow $(G^0 \to G)$.

Since D^0 contains an arrow with initial vertex $G^0 = C_{p_1p_2\ldots p_{n-d}}$, there are exactly $n-d$ bottom faces which do not intersect D^0, namely

$$B_1, B_2, \ldots, B_{n-d}.$$

We need to show that any arrow of D is either an arrow of D^0, or contained in one of the bottom facets

$$B_1, B_2, \ldots, B_{n-d}.$$

Assume by contradiction that it is not the case. Note that an arrow is not in D^0 if and only if its initial vertex does not contain G_0, and an arrow is not in one of the bottom facets

$$B_1, B_2, \ldots, B_{n-d}$$

if and only if its final vertex is not in the union of subgroups $\bigcup_{i=1}^{n-d} C_{p_1\ldots\hat{p}_i\ldots p_n}$. Therefore, D must have an arrow $(K \to L)$ with $K \not\supset G^0$ and

$$L \not\subset \bigcup_{i=1}^{n-d} C_{p_1\ldots\hat{p}_i\ldots p_n}.$$

The latter condition implies that L must contain the subgroup $C_{p_1\cdots p_{n-d}} = G^0$. By the restriction condition, since D contains the arrow $(K \to L)$ it must also contain the arrow

$$(K \cap G^0 \to L \cap G^0 = G^0).$$

But we saw that D contains the arrow $(G^0 \to G)$, so by transitivity D must contain the arrow

$$(K \cap G^0 \to G).$$

Since $K \not\supset G^0$, the subgroup $K \cap G^0$ is a strict subset of G^0 with an arrow to G, which contradicts the minimality of G^0. Therefore, any arrow of D is either an arrow of D^0, or contained in one of the bottom facets

$$B_1, B_2, \ldots, B_{n-d},$$

which concludes the proof. Q.E.D.

4.2 An involution of N_∞-diagrams for $G = C_{p_1\cdots p_n}$

In this section we introduce an involution $\Phi_n : \mathcal{N}_n \to \mathcal{N}_n$, which swaps the distinguished subsets $\mathrm{Comp}_0(G)$ and $\mathrm{Comp}_n(G)$. We construct the involution by induction on $n \geq 1$ as follows.

- If $n = 1$, then we let Φ_1 swap the empty N_∞-diagram and the full N_∞-diagram.
- Now assume that we constructed the map Φ_n for a fixed $n \geq 1$, and consider a N_∞-diagram D on the $(n+1)$-dimensional cube, i.e., for

  $$G = C_{p_1\cdots p_{n+1}}.$$

  To define Φ_{n+1}, we apply Φ_n to D restricted to each facet and we reindex the vertices so that in the image a vertex H is replaced with G/H (which requires also reversing the direction of the arrows). In particular this means that Φ_{n+1} sends a bottom facet B_i to the top facet T_i, and vice-versa. Finally, the big diagonal $(1 \to G)$ belongs to $\Phi_{n+1}(D)$ if and only if $D \in \mathrm{Comp}_0$.

We claim that this construction gives a well-defined map from the set of N_∞-diagrams to the set of graphs on the hypercube. Later we will prove that the image of a N_∞-diagram is also a N_∞-diagram.

Proposition 4.6. For any $n \geq 1$ the map Φ_n is a well-defined map from the set $\mathcal{N}_n$ of N_∞-diagrams on the n-dimensional hypercube to the set of graphs on the hypercube.

Proof. First notice that Φ_n acts as the complement on cube edges, with vertices swapped as follows:

$$H \mapsto G/H.$$

Thus, it is well-defined on cube edges, and we only need to check that it is well-defined on all other arrows. If $n = 1$ all arrows are cube edges, so there is nothing to check. Now assume Φ_m is well-defined for $m \leq n$, and consider a N_∞-diagram D on the $(n+1)$-dimensional hypercube. The induction hypothesis shows that $\Phi_{n+1}(D)$ is well-defined on all the diagonals of the form $(H \to K)$ when $H \neq 1$ or $K \neq G$. Indeed, those are diagonals of smaller hypercubes. So we only need to consider the instructions for definining Φ_n on the big diagonal $(1 \to G)$, which is clear. Q.E.D.

We now prove that the image of an N_∞-diagram is an N_∞-diagram.

Theorem 4.7. For any N_∞-diagram $D \in \mathcal{N}_n$ we have $\Phi_n(D) \in \mathcal{N}_n$.

Proof. The case $n = 1$ is clear. Now assume $\Phi_m(\mathcal{N}_m) \subseteq \mathcal{N}_m$ for $m \leq n$, and consider a N_∞-diagram D on the $(n+1)$-dimensional hypercube. We need to prove the following two properties:

- (restriction condition) for any arrow $(H \to K)$ in $\Phi_{n+1}(D)$ and any subgroup L of G such that $H \cap L \neq K \cap L$, the arrow $(H \cap L \to K \cap L)$ is also in $\Phi_{n+1}(D)$;
- (transitivity condition) for any arrows $(H \to K)$ and $(K \to L)$ in $\Phi_{n+1}(D)$, the arrow $(H \to L)$ is also in $\Phi_{n+1}(D)$.

Let us first check that Φ_{n+1} preserves the restriction condition. Let $(H \to K)$ be any arrow which is not the big diagonal. Then, by induction on the diagram D_H^K, which is defined on a smaller cube, we immediately see that the restriction condition is satisfied for all arrows, except maybe the big diagonal. So we may assume that D is such that the arrow $(1 \to G)$ belongs to $\Phi_{n+1}(D)$, and we need to show that the arrows $(1 \to L)$ for any subgroup L of G also belong to $\Phi_{n+1}(D)$. By definition of $\Phi_{n+1}(D)$ we know that D has no arrow adjacent to the vertex G. Let $K := G/L$, and consider the induced diagram D_K^G, which has no arrows to G either. By the induction hypothesis it follows that its image by Φ contains the 'big' diagonal $(1 \to G/K)$, that is, $(1 \to L)$. Therefore, $(1 \to L)$ belongs to $\Phi_{n+1}(D)$ as claimed.

Let us now look at the transitivity condition, i.e., let us check that for any arrow $(H \to K)$ and $(K \to L)$ in $\Phi_{n+1}(D)$, the arrow $(H \to L)$ is also in $\Phi_{n+1}(D)$. By induction on a smaller cube, it is immediate to see that the transitivity condition holds when $H \neq 1$ or $L \neq G$. So we only need to prove transitivity for arrows $(1 \to K)$ and $(K \to G)$.

First, note that the restriction condition on the (smaller) diagram D_1^K implies that all arrows $(1 \to H)$ are in $\Phi_{n+1}(D)$ if $H \subset K$. So let us consider a subgroup $H \neq G$ such that $H \cap K \neq H$. By the restriction condition, since the arrow $(K \to G)$ is in $\Phi_{n+1}(D)$, so is the arrow $(H \cap K \to H)$. Since $H \cap K \subset K$, the arrow $(1 \to H \cap K)$ is in $\Phi_{n+1}(D)$. By transitivity in the diagram D_1^H

it follows that the arrow $(1 \to H)$ is in $\Phi_{n+1}(D)$. So we have proved that $\Phi_{n+1}(D)$ contains all arrows $(1 \to H)$, except maybe if $H = G$.

Now we are left with checking that $\Phi_{n+1}(D)$ contains the big diagonal. Indeed, if it did not, then by definition of Φ_{n+1} it would mean that D has an arrow $(L \to G)$ for some subgroup L. This would imply that $\Phi_{n+1}(D)$ has no arrow $(1 \to G/L)$, which contradicts the fact $\Phi_{n+1}(D)$ must contain all arrows $(1 \to H)$ for $H \neq G$. So $\Phi_{n+1}(D)$ contains the big diagonal, which concludes the proof that Φ_{n+1} preserves the transitivity condition. Q.E.D.

Proposition 4.8.

1. The map Φ_n interchanges the subsets $\mathrm{Comp}_0(G)$ and $\mathrm{Comp}_n(G)$.
2. The map Φ_n is an involution.
3. $\Phi_n(\mathrm{Comp}_d(G)) = \mathrm{Comp}_{n-d}(G)$ for any $0 \leq d \leq n$.

Proof.

1. This is immediate from the construction of Φ_n.
2. Recall from the proof of Proposition 4.6 that Φ_n acts as the complement on cube edges, with vertices swapped as follows: $H \mapsto G/H$. Therefore Φ_n^2 acts as the identity on cube edges.

   We need to show that it acts the same way on diagonals. If $n = 1$, this is clear. Now assume $n \geq 2$ and consider $D \in \mathcal{N}_n$. By induction, Φ^2 is the identity on all diagonals, except maybe the big diagonal $(1 \to G)$.

   Assume first $(1 \to G) \in D$, i.e $D \in \mathrm{Comp}_n$. Then by (1) we get that $\Phi_n(D) \in \mathrm{Comp}_0$, and again that $\Phi_n^2(D) \in \mathrm{Comp}_n$, so that $(1 \to G) \in D$. Now if $(1 \to G) \notin D$, then $D \notin \mathrm{Comp}_n$. Then by (1) we get that $\Phi_n(D) \notin \mathrm{Comp}_0$, and again that $\Phi_n^2(D) \notin \mathrm{Comp}_n$, so that $(1 \to G) \notin D$.
3. Consider a diagram $D \in \mathrm{Comp}_d(G)$ for $G = C_{p_1 \cdots p_n}$. Without loss of generality we may assume that $G^0 = C_{p_1 \dots p_{n-d}}$, and we know from Proposition 4.5 that arrows in D are contained in the union of the facets $B_1, \dots, B_{n-d}$ and of $D^0 = D^G_{G^0}$. We also know that D contains the arrow $(G^0 \to G)$. Let $E := \Phi_n(D)$. We claim that $E^0 = E^G_{G/G^0}$ and that the arrows of E are either arrows of E^0 or contained in the bottom facets $B_{n-d+1}, \dots, B_n$. By definition, this would imply that $E \in \mathrm{Comp}_{n-d}(G)$.This is equivalent to proving that E contains the arrow $(G/G^0 \to G)$, and contains no arrow $(K \to L)$ with $L \supset G/G^0$ and $K \not\supset G/G^0$.

   Consider the induced diagram $D_1^{G^0}$. In this diagram, there are no arrows adjacent to G^0. Indeed, such an arrow would not be contained in D^0, nor in the union of facets $B_1, \dots, B_{n-d}$. Therefore, the image E^0 of $D_1^{G^0}$ by Φ_n must contain the big diagonal of E^0, namely, the arrow
   $$(G/G^0 = C_{p_{n-d+1} \cdots p_n} \to G).$$

   Now assume by contradiction that it contains an arrow $(K \to L)$ with $L \supset G/G^0$ and $K \not\supset G/G^0$. Let $M := K \cap G/G^0$. As in the proof of Proposition 4.5, we deduce that E contains an arrow $(M \to G)$. Therefore, D has no arrow to G/M. Now note that G/M

strictly contains G^0. Since D contains the arrow $(G^0 \to G)$, the restriction condition implies that D also contains the arrow

$$(G^0 \cap G/M = G_0 \to G/M),$$

which is a contradiction. This concludes the proof. Q.E.D.

Corollary 4.9. Let $G = C_{p_1 \cdots p_n}$, then

$$|\mathcal{N}_n| = \begin{cases} \sum_{i=0}^{(n-1)/2} 2 \times |\operatorname{Comp}_i(G)| & \text{if } n \text{ is odd,} \\ \sum_{i=0}^{n/2-1} 2 \times |\operatorname{Comp}_i(G)| + |\operatorname{Comp}_{n/2}(G)| & \text{if } n \text{ is even.} \end{cases}$$

5 Enumerating diagrams for $G = C_{pqr}$

We will now use the results of the previous section to enumerate the number of N_∞-operads for $G = C_{pqr}$. Using a code (which does not make use of any additional structure), we have calculated that there are 450 such, however, we will now show this using the theory as opposed to naive computational effort. From Proposition 4.8 we know that it is enough to compute the cardinalities of $\operatorname{Comp}_0(C_{pqr})$ and $\operatorname{Comp}_1(C_{pqr})$, and then $|\mathcal{N}_3| = 2(|\operatorname{Comp}_0(C_{pqr})| + |\operatorname{Comp}_1(C_{pqr})|)$.

Lemma 5.1. The size of $\operatorname{Comp}_3(C_{pqr})$ is 198.

Proof. Recall that $\operatorname{Comp}_0(C_{pqr}) = \operatorname{Comp}_3(C_{pqr})$ consists of all N_∞-diagrams containing the arrow $(1 \to C_{pqr})$ and therefore, by restriction, all arrows $(1 \to H)$ for all $H < C_{pqr}$. Therefore, we must count the possibilities of filling in the three two-dimensional facets containing C_{pqr} in a manner that gives an N_∞-diagram. We distinguish the different cases according to how many "edge" arrows $(C_{ij} \to C_{pqr})$ there are in an N_∞-diagram before considering the possibilities for the three top facets. We will then view the N_∞-diagram on a top facet as an N_∞-diagram for the two-dimensional case with top vertex C_{pq}, as we will also see in the figures that follow.

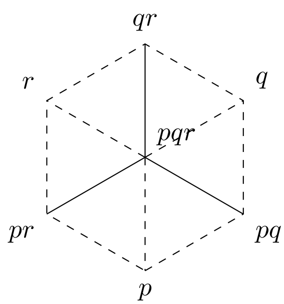

FIGURE 7. The three facets of the three dimensional cube containing C_{pqr}. The arrows $(C_{ij} \to C_{pqr})$, which are each shared by two facets, are indicated with solid lines.

Case 1. There are no arrows connecting to the vertex C_{pqr} from a C_{ij}. This restricts the N_∞-diagrams that could occur on the three top facets to those which do not contain the top arrows

$(C_p \to C_{pq})$ or $(C_q \to C_{pq})$ when considered as N_∞-diagrams for the two-dimensional case. There are five such N_∞-diagrams (Fig. 8.) and three faces to fill, thus $5^3 = 125$ remaining options.

$$\begin{tikzcd} 1 & q \\ p & pq \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[d] & q \\ p & pq \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r] & q \\ p & pq \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r] \arrow[d] & q \\ p & pq \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & q \\ p & pq \end{tikzcd}$$

FIGURE 8. The two dimensional N_∞-diagrams which can occur in a facet which contains no top arrows.

Case 2. There is one arrow $(C_{ij} \to C_{pqr})$. Thus, one of the three top facets contains no top arrows $(C_p \to C_{pq})$ or $(C_q \to C_{pq})$ while the other two contain one arrow of those two. There are two N_∞-diagrams of C_{pq} satisfying the latter condition (Fig. 9.), and there is a three-fold rotational symmetry, thus we have $3 \cdot 5 \cdot 2^2 = 60$ remaining options.

$$\begin{tikzcd} 1 \arrow[r] & q \\ p \arrow[r] & pq \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & q \\ p \arrow[r] & pq \end{tikzcd}$$

FIGURE 9. The two dimensional N_∞-diagrams which can occur in a facet which contains one top arrow (thicker line).

Case 3. There are two arrows of the form $(C_{ij} \to C_{pqr})$ connecting to the vertex C_{pqr}. Thus, one of the three top facets contains both $(C_p \to C_{pq})$ and $(C_q \to C_{pq})$ while the other two facets contain one such arrow. There is only one N_∞-operad of C_{pq} satisfying the former condition (Fig. 10.), and there is a three-fold rotational symmetry, thus we have $3 \cdot 2^2 = 12$ remaining options.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr] & q \arrow[d] \\ p \arrow[r] & pq \end{tikzcd}$$

FIGURE 10. The two-dimensional N_∞-diagrams which can occur in a facet which contains both top arrows (thicker lines).

Case 4. All three arrows connecting to the vertex C_{pqr} are present. Thus all three top facets must contain two arrows $(C_p \to C_{pq})$ and $(C_q \to C_{pq})$, and so there is only one remaining option.

These cases are disjoint and account for all possible N_∞-diagrams, and the possibilities contained therein sum to 198. Q.E.D.

Lemma 5.2. The size of $\mathrm{Comp}_1(C_{pqr}) = \mathrm{Comp}_2(C_{pqr})$ is 27.

Proof. Without loss of generality, the only arrow adjacent to C_{pqr} is $(C_{pq} \to C_{pqr})$. By restriction, our N_∞-diagram therefore also contains the parallel edges $(C_q \to C_{qr})$, $(C_p \to C_{pr})$ and $(C_1 \to C_r)$.

The only other arrows that could occur are in the facets containing 1 and C_{pr} and 1 and C_{qr}. We distinguish the possible cases according to how many of those facets contain a diagonal arrow $(C_1 \to C_{pr})$ or $(C_1 \to C_{qr})$.

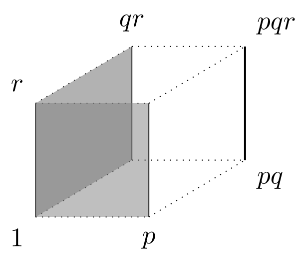

FIGURE 11. The arrows induced by restriction (solid lines) if $(C_{pr} \to C_{pqr})$ is present (thicker line). An N_∞-diagram containing $(C_{pr} \to C_{pqr})$ in $\mathrm{Comp}_1(C_{pqr})$can only include additional arrows in the gray facets.

Case 1. None of the two facets contain a diagonal. There is only one such case.

Case 2. One facet contains a diagonal, and the other one does not. The one that does not contain a diagonal can therefore contain no further arrows, whereas the other one is one of the two possible diagrams in Figure 12. This therefore accounts for four possibilities.

Case 3. Both facets contain their diagonal. Therefore, each of them is one of the two diagrams in Figure 12. This gives us four possibilities.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

FIGURE 12. The possible forms of an N_∞-diagram containing the diagonal and two parallel arrows (thicker lines).

In all, we see that we have nine possibilities for N_∞-diagrams in $\mathrm{Comp}_1(C_{pqr})$ containing the arrow $(C_{pq} \to C_{pqr})$, so also nine for $(C_{pr} \to C_{pqr})$ and nine for $(C_{qr} \to C_{pqr})$. Q.E.D.

Corollary 5.3. $|\mathcal{N}_3| = 198 \times 2 + 27 \times 2 = 450$.

Proof. By Corollary 4.9, we have $|\mathcal{N}_3| = |\operatorname{Comp}_3(C_{pqr})| \times 2 + |\operatorname{Comp}_1(C_{pqr})| \times 2$. Q.E.D.

Remark 5.4. We finish this paper by the consideration for $\mathcal{N}_n$ for $n > 3$. Although we have presented a way of decomposing the problem into enumerating $\lceil n/2 \rceil$ disjoint pieces, the way forward is still not clear. Indeed, the reasoning to get the values of 198 and 27 in the $n = 3$ case required studying in depth the cases appearing for $n = 2$. Therefore even for $n = 4$, one would have to be able to analyse the 450 options for $n = 3$ on a case-by-case basis. Computationally, we know that $\mathcal{N}_4 = 5,389,480$.

As such, the results appearing in this paper should be seen as a structural result as opposed to an algorithm for computation.

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